J. Appl. Maths Mechs Vol. 56, No. 3, pp. 319–323, 1992
Printed in Great Britain.

0021–8928/92 \$15.00 + .00

MONOTONICITY PROPERTIES OF CERTAIN TWO-DIMENSIONAL INCOMPRESSIBLE VORTEX FLOWS AND SUBSONIC GAS FLOWS†

A. I. RYLOV

Novosibirsk

(*Received* 1 *July* 1991)

It is proved that the pressure varies monotonically along certain sections of bodies and boundaries of flow regions, and that the isobars in the flow around convex bodies have no internal branch points (saddle points). The method used is the method of isobars, which is based on the fact that the angle of inclination of the velocity vector varies monotonically along curves of constant pressure (isobars) [1]. This method has been used before to study subsonic vortex flows between a body situated in a supersonic flow and an attached or receding shock wave [1–3].

1. CONSIDER a two-dimensional symmetrical subsonic vortex flow of gas or incompressible liquid around a body ab (see Fig. 1). Viscosity and thermal conduction are ignored and the gas is assumed to be polytropic. Throughout, we shall consider only the upper half of the flow, which is symmetrical about the x axis. The flow is from left to right, with the flow parallel to the x axis at infinity on the left. The pressure p is constant across the flow, but the density ρ and the magnitude of the velocity vector in vortex flow may depend on the stream function ψ. The flow is assumed to be without separation and without regions with closed streamlines. Finally, when the body ab is in a flow of gas we will assume that the Mach number M is less than 1 at all points of the region, i.e. there are no local supersonic zones.

Let us consider a straight segment cd on the body, where the inclination of the wall $\theta = \theta^+$ is a maximum for the body (more precisely, for its upper half), and a straight segment fg with minimum wall inclination $\theta = \theta^-$. The lengths of these segments may be zero, in which case we will be concerned with points at which the wall inclination is a maximum or a minimum.

For flows satisfying these conditions, one can prove the following theorem.

Theorem 1. The pressure p decreases (increases) monotonically along the segment of maximum (minimum) inclination.

Indeed, suppose that at some point t of cd the pressure is increasing, i.e. the acceleration at t is negative. Consider the isobar leaving the point t. The derivative of the angle of inclination of the velocity vector computed along the isobar is [1]

$$\theta_l = -p_n (1 - M^2 \sin^2 \beta)/(\rho q^2)$$

where β is the angle between the velocity vector and the isobar and p_n is the derivative of p along the normal to the isobar (as a point moves along the isobar, the normal points to the left).

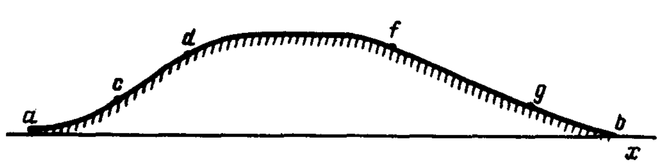


FIG. 1.

† *Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 386–391, 1992.

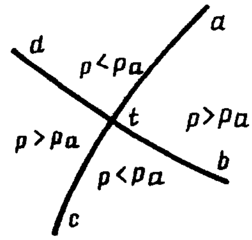


FIG. 2.

Following [1–3], making allowance for the existence of branch points [4], we will henceforth understand an isobar to be a curve $p =$ const bounding the region in which the pressure is higher or lower than at the boundary. That is to say: the continuation of an isobar after a branch point will be the branch adjoining the region indicated. For example, in Fig. 2, the isobar *at* of $p = p_a$, on passing through the branch point t, at which $p_n = 0$, must be continued by the branch *tb* (*td*) if the bounded region is one of higher (lower) pressure, but never by *tc*.

With isobars defined in this way, the derivative p_n does not change sign as the representative point moves along an isobar, so that if $M \leq 1$ the angle θ will vary monotonically along the isobar. This is also true for an incompressible liquid, for which the above ratio will have the following form: $\theta_l = -p_n/(\rho q^2)$.

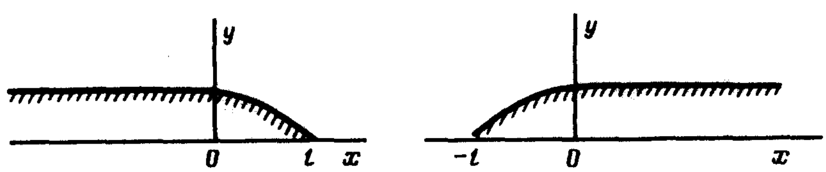
Taking this into account, we see that along the isobar that leaves a point t the angle θ increases monotonically. Consequently, the isobar cannot reach *ab* or the axis of symmetry, on both of which $\theta \leq \theta^+$; neither can it go to infinity, where $\theta = 0$. This contradiction proves the first part of the theorem. The proof that the pressure increases monotonically along the minimum angle segment is analogous. This proves the theorem.

Note that we are excluding a situation in which the angle θ may vary by an appreciable amount $\varphi \leq 2\pi$ along a given isobar, which would permit the isobar to hit the body or the boundary of the flow region. Analysis of the curves $\theta =$ const will show that this is possible only if there exist regions with closed streamlines and interior stagnation points [2]. The former have already been excluded explicitly, the latter are forbidden by the conditions at infinity to the left. Indeed, if there were an interior stagnation point between two streamlines, which converged there from infinity on the left, there would necessarily exist a streamline leaving the stagnation point and going left to infinity. This would contradict the assumption that $\theta = 0$ at infinity on the left.

Theorem 1 clearly remains valid if the flow is bounded by a wall at which $\theta^- \leq \theta \leq \theta^+$ (flow in a symmetrical duct).

In particular, application of Theorem 1 to a semi-infinite body generated by straight lines implies the following conclusions. At rear contours (Fig. 3a) where $\theta = 0$ for $x \leq 0$, $\theta \leq 0$ for $\theta \leq x \leq 1$, along a straight wall and along the axis of symmetry at $x \geq 1$, the flow accelerates. At forward contours (Fig. 3b, where $\theta = 0$ for $x \geq 0$, $\theta \geq 0$ for $-1 \leq x \leq 0$), along a straight wall and along the axis of symmetry at $x \leq -1$, the flow decelerates.

2. We will now consider some conditions on the shape of the body under which the isobars for symmetrical flow around the body will have no branch (saddle) points. The question of whether



(a) (b)

FIG. 3.

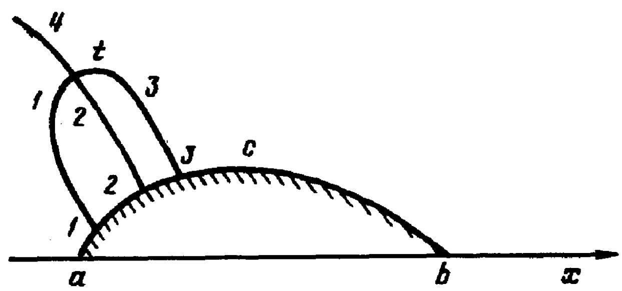


FIG. 4.

various singular points do or do not exist is of interest in the theory of liquid and gas flows in other, related branches of mathematical physics. Such points are, in particular, those at which both first derivatives p_x and p_y vanish, that is, either points where the pressure is a maximum or branch (saddle) points of isobars. In subsonic vortex flows, the first case has been investigated by the method of isobars [1], and it has been shown that the pressure cannot have an extremum at interior points of the flow. The only exceptions are flows with interior stagnation points and closed streamlines—e.g. in asymmetrical flow around a cylinder with high circulation numbers. Later we shall present an example of conditions on the shape of the body under which the isobars cannot have branch (saddle) points.

Theorem 2. In symmetrical flow around a convex body, along whose walls the angle θ cannot increase, the isobars can have no branch points inside the flow region (except at the walls of the body and on the axis of symmetry).

Proof. Let us consider the upper half of the convex body ab (Fig. 4). The angle θ does not increase along ab; along ac we have $\theta \geq 0$, and along cb—$\theta \leq 0$. Let t be a branch point of isobars. This means, in particular, that the first derivatives of p and θ with respect to x and y vanish there. Depending on the signs of the higher-order derivatives of p at t, an even number of isobars (at least four) converge at t, some possibly touching one another, and the same number of regions (bounded by these isobars) meet there. As one describes a circle around t, regions with higher pressure than at t alternate with regions of lower pressure. (A situation in which an isobar is the common boundary of two regions, each with pressure either higher or lower than along the isobar, cannot occur, because when $M<1$ the equality $p_n = 0$ may hold only at isolated points.) Accordingly, any isobar along which θ increases as the distance from t increases is followed (in the above sense, as a circle is described around the branch point) by an isobar along which θ decreases.

Let us assume, to fix our ideas, that $\theta = \theta_t \geq 0$ at t (the case $\theta_t < 0$ is treated similarly). Consider two nearby isobars along which θ increases as the distance from t increases (numbered 1 and 3 in Fig. 4). According to the boundary conditions, these isobars may hit the body only at the segment ac of the wall along which $\theta \geq 0$. At the points of contact—also numbered 1 and 3—$\theta_1 \geq \theta_t \geq 0$, $\theta_3 > \theta_t \geq 0$. Since the body is convex, $\theta_1 > \theta_3 > \theta_t \geq 0$. Between the above-mentioned isobars there must be another isobar, also hitting ac, along which θ decreases as the distance from t increases. This isobar and its point of contact are numbered 2 in Fig. 4. Thus, at point 2, between points 1 and 3, $\theta = \theta_2 < \theta_t < \theta_3$. But $\theta \geq \theta_3$ on the convex wall between points 1 and 3, which is a contradiction. This completes the proof of the theorem.

For convex semi-infinite bodies along which $\theta \geq 0$, such as that illustrated in Fig. 3(b), it is easy to prove that there are no branch points on the axis of symmetry either.

3. We will now consider a uniform two-dimensional symmetric vortex-free subsonic flow of gas around a wedge of finite thickness touching a horizontal wall at its tip (Fig. 5). An important feature of this flow is that, however low the (subsonic) velocity of the oncoming flow, a local supersonic zone (LSZ) is formed in the neighbourhood of the tip c; its boundary is the sonic line emanating from c and a closing shock wave (CSW)—represented in Fig. 5 by a dashed curve and a solid curve,

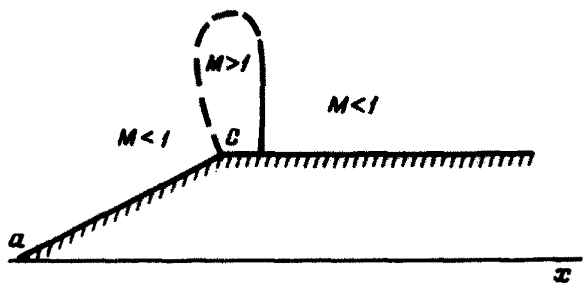


FIG. 5.

respectively. Note that the flow in the LSZ is fairly complex. Besides a fan-shaped region of rarefaction, it may also contain internal shock waves other than the CSW. For our purposes the important point is that the boundary of the LSZ contains part of a shock wave.

As it turns out, the fact that the pressure is monotonic along the wedge wall ac is related directly to the value of the angle θ along the CSW. We have the following theorem.

Theorem 3. If θ does not exceed the angle of the wedge θ_0 on the CSW (outside the LSZ), then the pressure p decreases monotonically along the wedge wall ac (Fig. 5).

Indeed, suppose that the pressure increases at some point t on ac. Then θ increases along the isobar emanating from t. The Mach number does not increase along the isobar, since the entropy does not decrease, and the overall pressure does not increase. Consequently, the isobar can do only one thing—hit the CSW. But in that case $\theta > \theta_0$ at an appropriate point of the CSW, contrary to our assumption.

Even lacking rigorous bounds for θ on the CSW, it is clearly quite unlikely that $\theta > \theta_0$ on the CSW. Consequently, it is equally unlikely that p will vary non-monotonically on ac. This has been confirmed by experiment [5], as only monotone variation of p has been observed along the wall of the wedge.

In the case of flat contact of a finite wedge with a horizontal wall, an LSZ may exist whose boundary contains no shock segments. In that case one can prove rigorously that the flow is monotonic along the wall.

4. Let us consider a two-dimensional vortex-free flow of gas in a symmetric Laval nozzle, whose converging section ab (Fig. 6), where $\theta \leq 0$, is preceded by an infinite straight wall along which $\theta = 0$. We shall assume that the converging section is sufficiently smooth, so that to the left of the sonic line cd we have $M < 1$.

By the Nikol'skii–Taganov Theorem [6], $\theta < 0$ at the point c. We can prove the following theorem.

Theorem 4. Along the infinite horizontal wall, the pressure increases monotonically up to the point a; along the axis of symmetry, up to the sonic point d, it decreases monotonically.

Indeed, suppose that at some point t on the straight wall the flow accelerates. Then θ will increase along the isobar leaving that point. Hence the isobar cannot reach as far as the sonic line, on which the pressure is less than at t, or to the axis of symmetry or straight wall, on which $\theta = 0$, or to the wall ac, where $\theta < 0$.

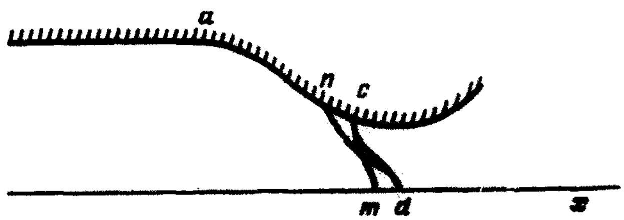


FIG. 6.

Similarly one can prove that the pressure decreases monotonically along the axis of symmetry to the left of d, completing the proof of the theorem.

With certain reservations, the theorem carries over to vortex flows. There are two possible cases.

1. The derivative of the overall pressure is non-negative: $dp_0/d\psi \geqslant 0$, where ψ is the stream function, defined in the usual way. Then $M \leqslant 1$ on the isobar cm emanating from the sonic point c of the upper contour. It can be shown that Theorem 3 remains valid for the straight upper wall to the left of a and for the axis of symmetry to the left of m. Isobars issuing from the segment md can only reach the sonic line cd. Consequently, if the pressure is monotonic along cd, it is also monotonic along md. In turn, we observe that when $dp_0/d\psi \geqslant 0$ the pressure p will vary monotonically along cd if each streamline cuts cd only once.

2. $dp_0/d\psi \geqslant 0$. In that case $M \leqslant 1$ on the isobar dn emanating from the point d, and $\theta < 0$ at the point n. Under these conditions Theorem 3 remains completely valid.

We note that there is a certain analogy between Theorem 1 above and some results for two-dimensional irrotational jet flows, for which it has been proved that θ varies monotonically along the free boundary p = const when p takes extremal values on that boundary [7]. Essentially, what happens in these results for jets and in Theorem 1 is that p and θ change places.

We could have used the maximum principle for θ in this paper. However, it should be borne in mind that the maximum principles for p and θ in subsonic vortex flows of a gas and an incompressible liquid were established by analysing curves p = const and θ = const along which θ and p vary monotonically [1]. In addition, even had we used the maximum principle for θ in Theorem 4, it would nevertheless have been necessary to analyse the isobars in order to find bounds for θ on some segments of the boundaries of the regions.

The authors is indebted to P. I. Plotnikov for useful comments.

REFERENCES

1. NIKOL'SKII A. A., Two-dimensional vortex flows of gas. In A. A. NIKOL'SKII, *Theoretical Investigations in the Mechanics of Liquid and Gas*. Trudy TsAGI, No. 2122, pp. 74–85, 1981.
2. RYLOV A. I., Possible regimes of flow around sharp bodies of finite thickness at arbitrary supersonic velocities of the oncoming stream. *Prikl. Mat. Mekh.* **55,** 95–99, 1991.
3. RYLOV A. I., Some properties of subsonic flow after a shock wave arising in supersonic flow around bodies of finite thickness. *Prikl. Mat. Mekh.* **55,** 780–786, 1991.
4. MISES R. VON, *Mathematical Theory of Compressible Fluid Flow*. Academic Press, New York, 1958.
5. SAKHNENKO T. M., Experimental investigation of the transonic flow of a gas around wedges. *Uch. Zap. TsAGI* **21,** 111–114, 1990.
6. NIKOL'SKII A. A. and TAGANOV G. I., The motion of a gas in the local supersonic zone and certain conditions in which potential flow is violated. *Prikl. Mat. Mekh.* **10,** 481–502, 1946.
7. BIRKHOFF G. and ZARANTONELLO E. H., *Jets, Wakes and Cavities*. Academic Press, New York, 1957.

Translated by D.L.